

13.2

Limits of Quotients

Find the limits in Exercises 13–24 by rewriting the fractions first.

$$20. \lim_{\substack{(x,y) \rightarrow (4,3) \\ x \neq y+1}} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}$$

Sol'n: $\frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1} \cdot \frac{\sqrt{x} + \sqrt{y+1}}{\sqrt{x} + \sqrt{y+1}} = \frac{\cancel{x} - \cancel{y} - 1}{(\cancel{x} - \cancel{y} - 1)(\sqrt{x} + \sqrt{y+1})} = \frac{1}{\sqrt{x} + \sqrt{y+1}}$

So $\lim_{\substack{(x,y) \rightarrow (4,3) \\ x \neq y+1}} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1} = \lim_{\substack{(x,y) \rightarrow (4,3) \\ x \neq y+1}} \frac{1}{\sqrt{x} + \sqrt{y+1}} = \frac{1}{\sqrt{4} + \sqrt{3+1}} = \boxed{\frac{1}{4}}$

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Limits with Three Variables

Find the limits in Exercises 25–30.

25. $\lim_{P \rightarrow (1, 3, 4)} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)$

26. $\lim_{P \rightarrow (1, -1, -1)} \frac{2xy + yz}{x^2 + z^2}$

27. $\lim_{P \rightarrow (\pi, \pi, 0)} (\sin^2 x + \cos^2 y + \sec^2 z)$

28. $\lim_{P \rightarrow (-1/4, \pi/2, 2)} \tan^{-1} xyz$

29. $\lim_{P \rightarrow (\pi, 0, 3)} ze^{-2y} \cos 2x$

Soln: $\lim_{P \rightarrow (-\frac{1}{4}, \frac{\pi}{2}, 2)} \tan^{-1}(xyz) = \tan^{-1}\left(-\frac{1}{4} \cdot \frac{\pi}{2} \cdot 2\right) = \boxed{\tan^{-1}\left(-\frac{\pi}{4}\right)}$

13.2

Continuity for Three Variables

At what points (x, y, z) in space are the functions in Exercises 35–40 continuous?

38. a. $h(x, y, z) = \frac{1}{|y| + |z|}$ b. $h(x, y, z) = \frac{1}{|xy| + |z|}$

Sol'n: a) Note that h is not well-defined when $|y| = |z| = 0 \Rightarrow y = z = 0$.

Show using ε - δ def'n of continuity that h is continuous on $\{(x, y, z) \in \mathbb{R}^3 : y, z \neq 0\}$.

Let $(x_0, y_0, z_0) \in \mathbb{R}^3$ s.t. $y_0 \neq 0, z_0 \neq 0$. Let $\varepsilon > 0$. Then

$$\begin{aligned} |h(x, y, z) - h(x_0, y_0, z_0)| &= \left| \frac{1}{|y| + |z|} - \frac{1}{|y_0| + |z_0|} \right| \\ &= \left| \frac{|y_0| + |z_0| - |y| - |z|}{(|y| + |z|)(|y_0| + |z_0|)} \right| \end{aligned}$$

$$\leq \frac{|y| - |y_0|}{(|y| + |z|)(|y_0| + |z_0|)} + \frac{|z| - |z_0|}{(|y| + |z|)(|y_0| + |z_0|)} \quad (*)$$

Since $y_0 \neq 0, z_0 \neq 0$, $m = \min\{|y_0|, |z_0|\} > 0$. Then for $|y|, |z| \leq m$,

$$0 < 4m^2 = (2m)(2m) \leq (|y| + |z|)(|y_0| + |z_0|).$$

So (*) becomes

$$|h(x, y, z) - h(x_0, y_0, z_0)| \leq \frac{|y| - |y_0|}{4m^2} + \frac{|z| - |z_0|}{4m^2}.$$

Then for $\delta = 2m^2\varepsilon$ with $\sqrt{|x - x_0|^2 + |y - y_0|^2 + |z - z_0|^2} < \delta$, we have

$$|h(x, y, z) - h(x_0, y_0, z_0)| \leq \frac{|y| - |y_0|}{4m^2} + \frac{|z| - |z_0|}{4m^2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

And so h is continuous on the set $\{(x, y, z) \in \mathbb{R}^3 : y, z \neq 0\}$, i.e. $\mathbb{R}^3 \setminus x\text{-axis}$.

b) Similarly one can show that $\frac{1}{|xy|+|z|}$ is continuous on the set
 $\{(x,y,z) \in \mathbb{R}^3 : xy \neq 0 \text{ and } z \neq 0\}$ i.e. on $\mathbb{R}^3 \setminus (x\text{-axis} \cup y\text{-axis})$.

B.2

60. Does knowing that

$$2|xy| - \frac{x^2 y^2}{6} < 4 - 4 \cos \sqrt{|xy|} < 2|xy|$$

tell you anything about

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4 - 4 \cos \sqrt{|xy|}}{|xy|} ?$$

Give reasons for your answer.

Soln: Yes: We have

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4 - 4 \cos \sqrt{|xy|}}{|xy|} \leq \lim_{(x,y) \rightarrow (0,0)} \frac{2|xy|}{|xy|} = 2, \text{ and}$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4 - 4 \cos \sqrt{|xy|}}{|xy|} \geq \lim_{(x,y) \rightarrow (0,0)} \frac{2|xy| - \frac{x^2 y^2}{6}}{|xy|} = \lim_{(x,y) \rightarrow (0,0)} 2 - \frac{\sqrt{x^2 y^2}}{6} = 2.$$

So we can conclude by the Squeeze Theorem that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4 - 4 \cos \sqrt{|xy|}}{|xy|} = 2. \checkmark$$

13.2

In Exercises 65–70, find the limit of f as $(x, y) \rightarrow (0, 0)$ or show that the limit does not exist.

65. $f(x, y) = \frac{x^3 - xy^2}{x^2 + y^2}$

66. $f(x, y) = \cos\left(\frac{x^3 - y^3}{x^2 + y^2}\right)$

Sol'n: Polar coordinates: $x = r \cos \theta$, $y = r \sin \theta$, $x^2 + y^2 = r^2$.

$$\text{Then } f(x, y) = \cos\left(\frac{x^3 - y^3}{x^2 + y^2}\right) = \cos\left(\frac{r^3 \cos^3 \theta - r^3 \sin^3 \theta}{r^2}\right)$$

$$= \cos(r(\cos^3 \theta - \sin^3 \theta)) = f(r, \theta)$$

$$\text{and } \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{r \rightarrow 0} f(r, \theta) = \lim_{r \rightarrow 0} \cos(r(\cos^3 \theta - \sin^3 \theta))$$
$$= \cos(0) = \boxed{1}$$

13.2

In Exercises 71 and 72, define $f(0,0)$ in a way that extends f to be continuous at the origin.

71. $f(x, y) = \ln\left(\frac{3x^2 - x^2y^2 + 3y^2}{x^2 + y^2}\right)$

72. $f(x, y) = \frac{3x^2y}{x^2 + y^2}$

Soln: Polar coordinates: $x^2 + y^2 = r^2$, $x = r\cos\theta$, $y = r\sin\theta$.

$$\text{Then } f(x, y) = \frac{3x^2y}{x^2 + y^2} = \frac{3r^2\cos^2\theta r\sin\theta}{r^2} = 3r\cos^2\theta\sin\theta = f(r, \theta),$$

and

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{r \rightarrow 0} f(r, \theta) = \lim_{r \rightarrow 0} 3r\cos^2\theta\sin\theta = 0.$$

So defining $f(0,0) = 0$ extends f so that it is continuous at the origin. $\checkmark$

13.3

Calculating First-Order Partial Derivatives

In Exercises 1–22, find $\partial f/\partial x$ and $\partial f/\partial y$.

$$22. f(x, y) = \sum_{n=0}^{\infty} (xy)^n \quad (|xy| < 1)$$

Soln. $\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} \left(\sum_{n=0}^{\infty} (xy)^n \right) \overset{\text{since } |xy| < 1}{=} \sum_{n=0}^{\infty} \frac{\partial}{\partial x} (xy)^n = \sum_{n=1}^{\infty} n(xy)^{n-1} \frac{\partial}{\partial x} (xy)$

$$= \sum_{n=1}^{\infty} n(xy)^{n-1} y = \sum_{n=1}^{\infty} n x^{n-1} y^n$$

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} \left(\sum_{n=0}^{\infty} (xy)^n \right) \overset{\text{since } |xy| < 1}{=} \sum_{n=0}^{\infty} \frac{\partial}{\partial y} (xy)^n = \sum_{n=1}^{\infty} n(xy)^{n-1} \frac{\partial}{\partial y} (xy)$$

$$= \sum_{n=1}^{\infty} n(xy)^{n-1} x = \sum_{n=1}^{\infty} n x^n y^{n-1}$$

13.3

Calculating Second-Order Partial Derivatives

Find all the second-order partial derivatives of the functions in Exercises 41–54.

41. $f(x, y) = x + y + xy$ 42. $f(x, y) = \sin xy$

43. $g(x, y) = x^2y + \cos y + y \sin x$

44. $h(x, y) = xe^y + y + 1$ 45. $r(x, y) = \ln(x + y)$

46. $s(x, y) = \arctan(y/x)$ 47. $w = x^2 \tan(xy)$

Sol'n:

$$s_x = \frac{\partial}{\partial x} s(x, y) = \frac{\partial}{\partial x} \arctan\left(\frac{y}{x}\right) = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{\partial}{\partial x} \left(\frac{y}{x}\right) = \frac{-y x^{-2}}{1 + \left(\frac{y}{x}\right)^2} = \frac{-y}{x^2 + y^2}$$

$$s_y = \frac{\partial}{\partial y} s(x, y) = \frac{\partial}{\partial y} \arctan\left(\frac{y}{x}\right) = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \frac{\partial}{\partial y} \left(\frac{y}{x}\right) = \frac{x^{-1}}{1 + \left(\frac{y}{x}\right)^2} = \frac{x}{x^2 + y^2}$$

$$s_{xx} = \frac{\partial}{\partial x} \left(\frac{-y}{x^2 + y^2} \right) = \frac{(x^2 + y^2)(0) - (-y)(2x)}{(x^2 + y^2)^2} = \frac{2xy}{(x^2 + y^2)^2}$$

$$s_{yy} = \frac{\partial}{\partial y} \left(\frac{x}{x^2 + y^2} \right) = \frac{(x^2 + y^2)(0) - (x)(2y)}{(x^2 + y^2)^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$s_{xy} = \frac{\partial}{\partial y} \left(\frac{-y}{x^2+y^2} \right) = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{-x^2 - y^2 + 2y^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$s_{yx} = \frac{\partial}{\partial x} \left(\frac{x}{x^2+y^2} \right) = \frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

13.3

Mixed Partial Derivatives

In Exercises 55–60, verify that $w_{xy} = w_{yx}$.

55. $w = \ln(2x + 3y)$

56. $w = e^x + x \ln y + y \ln x$

Sol'n: $w_x = e^x + \ln y + \frac{y}{x}$

$$w_y = \frac{x}{y} + \ln x$$

$$w_{xy} = \frac{\partial}{\partial y} w_x = \frac{\partial}{\partial y} \left(e^x + \ln y + \frac{y}{x} \right) = \frac{1}{y} + \frac{1}{x}.$$

$$w_{yx} = \frac{\partial}{\partial x} w_y = \frac{\partial}{\partial x} \left(\frac{x}{y} + \ln x \right) = \frac{1}{y} + \frac{1}{x}.$$

So indeed $w_{xy} = w_{yx}$. ✓

13.3**Using the Partial Derivative Definition**

In Exercises 63–66, use the limit definition of partial derivative to compute the partial derivatives of the functions at the specified points.

$$66. f(x, y) = \begin{cases} \frac{\sin(x^3 + y^4)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0), \end{cases}$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y} \text{ at } (0, 0)$$

$$\begin{aligned} \text{Sol'n: } \frac{\partial f}{\partial x} \Big|_{(0,0)} &= \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{\sin(h^3 + 0^4)}{h^2 + 0^2} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{\sin(h^3)}{h^2}}{h} = \lim_{h \rightarrow 0} \frac{\sin(h^3)}{h^3} \xrightarrow{\text{L'Hopital's}} \lim_{h \rightarrow 0} \frac{3h^2 \cos(h^3)}{3h^2} = 1 \end{aligned}$$

$$\text{So } \frac{\partial f}{\partial x} \Big|_{(0,0)} = 1$$

$$\begin{aligned}
 \frac{\partial f}{\partial y} \Big|_{(0,0)} &= \lim_{h \rightarrow 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{\sin(h^4)}{h^2} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(h^4)}{h^3} \xrightarrow{\text{'Hôpital'}} \lim_{h \rightarrow 0} \frac{4h^3 \cos(h^4)}{3h^2} = \lim_{h \rightarrow 0} \frac{4}{3} h \cos(h^4) \\
 &= 0.
 \end{aligned}$$

$$\text{So } \frac{\partial f}{\partial y} \Big|_{(0,0)} = 0.$$